Continuous random variables

Probability density function

Cumulative distribution function

Common distributions

Continuous random variables

Uncountable number of values

Examples:

- Time taken to perform a task
- Weight of a parcel coming through a post office
- Remaining lifetime of a person
- Value of an investment fund

Probability density function

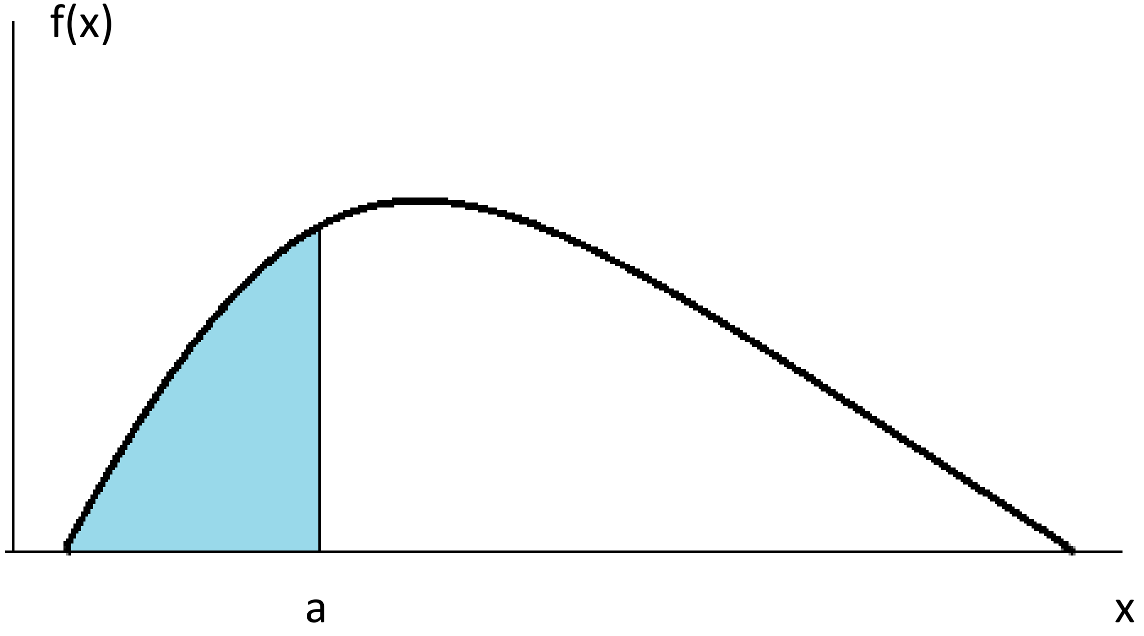
- Discrete random variable: probability mass function

P(X=x)

- Continuous random variable: probability density function

Probabilities are calculated by calculating area under the probability density function curve.

Probability density function



Total area under the curve = 1

Shaded area = P(X≤a) = F(a)

Probability density function

Properties of a probability density function:

- $f(x) \geq 0$
- $\int_{-\infty}^{\infty} f(x)dx = 1$

Probability density function

For a continuous random variable $P(X=a) = 0$, so we can only find probabilities that X lies in a certain interval.

So for a continuous random variable

$P(X<a) = P(X\leq a)$

Integration formulas

$$\int a\,dx = ax + c$$

$$\int x\,dx = \frac{1}{2}x^2 + c$$

$$\int x^2\,dx = \frac{1}{3}x^3 + c$$

$$\boldsymbol{\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c}$$

$$\int e^x\,dx = e^x + c$$

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

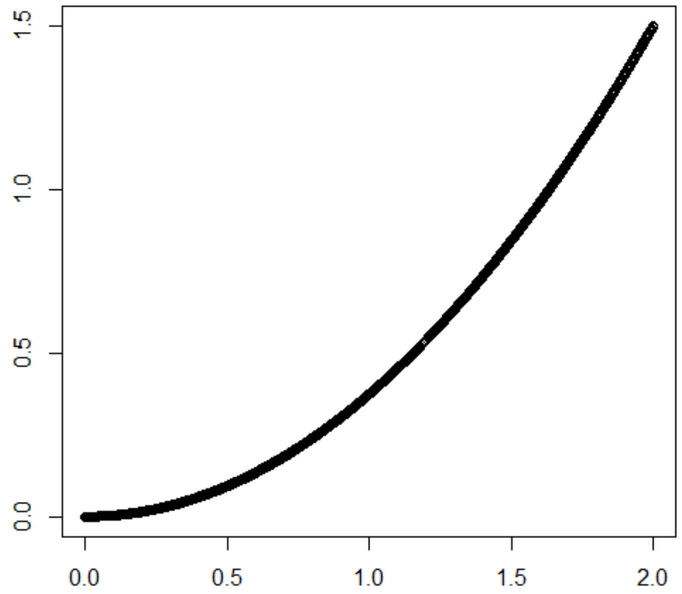
Integration

$$\int_0^1 x^5 \, dx = \left[\frac{1}{6}x^6\right]_0^1 = \frac{1}{6} \times (1^6 - 0) = \frac{1}{6}$$

$$\int_1^2 x^{-3} \, dx = \left[\frac{1}{-2}x^{-2}\right]_1^2 = -\frac{1}{2} \times \left[\frac{1}{x^2}\right]_1^2 = -\frac{1}{2} \times \left(\frac{1}{2^2} - \frac{1}{1^2}\right) = \frac{3}{8}$$

Example

$$f(x) = \begin{cases} \frac{3}{8}x^2 \; for \; 0 \le x \le 2 \\ 0 \qquad\qquad elsewhere \end{cases}$$



$$\int_0^2 \frac{3}{8}x^2 dx = \frac{3}{8} \times \left[\frac{1}{3}x^3\right]_0^2 = \frac{3}{8} \times \left(\frac{1}{3} \times 2^3 - \frac{1}{3} \times 0^3\right) = 1$$

Example

$$f(x) = \begin{cases} \frac{3}{8}x^2 \; for \; 0 \le x \le 2 \\ 0 \qquad\qquad elsewhere \end{cases}$$

$$P\left(X < \frac{1}{2}\right) = \int_0^{\frac{1}{2}} \frac{3}{8}x^2 dx = \frac{3}{8} \times \left[\frac{1}{3}x^3\right]_0^{\frac{1}{2}} = \frac{1}{8} \times \left(\left(\frac{1}{2}\right)^3 - 0^3\right) = \frac{1}{64}$$

$$P(X > 1) = \int_1^2 \frac{3}{8}x^2 dx = \frac{3}{8} \times \left[\frac{1}{3}x^3\right]_1^2 = \frac{1}{8} \times (2^3 - 1^3) = \frac{7}{8}$$

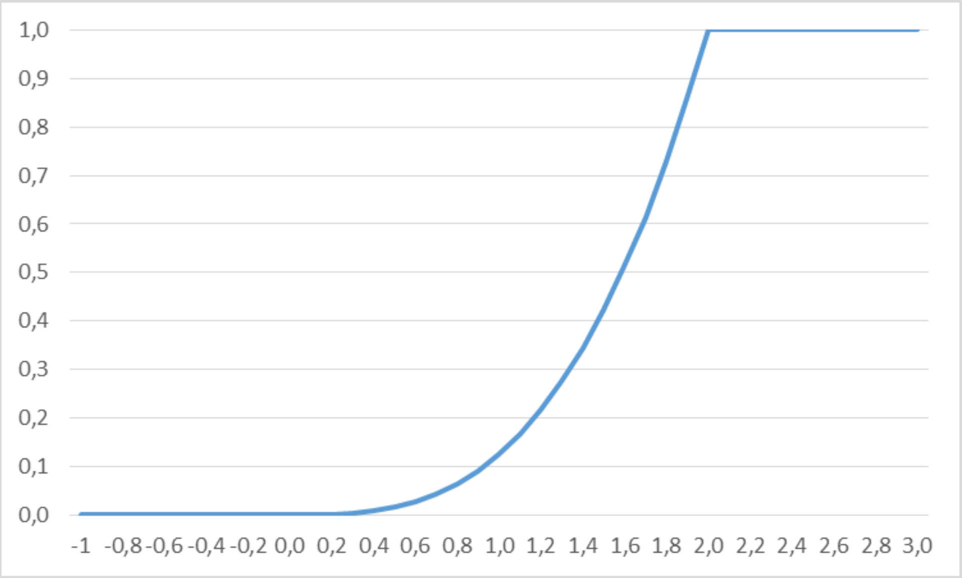
Example

Finding cumulative distribution function

$$F(a) = P(X \leq a) = \int_0^a \frac{3}{8}x^2 dx = \frac{3}{8} \times \left[\frac{1}{3}x^3\right]_0^a = \frac{1}{8} \times (a^3 - 0^3)$$

$$F(a) = \frac{a^3}{8}$$

$$F(x) = \begin{cases} 0 & for \quad x < 0 \\ \dfrac{x^3}{8} & for\, 0 \leq x \leq 2 \\ 1 & for \quad x > 2 \end{cases}$$



Example

$$f(x) = \begin{cases} \frac{3}{8}x^2 \ for\ 0 \leq x \leq 2 \\ 0 \qquad elsewhere \end{cases} \qquad F(x) = \begin{cases} 0 \quad for \quad x < 0 \\ \frac{x^3}{8} \quad for\, 0 \leq x \leq 2 \\ 1 \quad for \quad x > 2 \end{cases}$$

$$P(X < 1.2) = \int_0^{1.2} \frac{3}{8}x^2 dx = \frac{3}{8} \times \left[\frac{1}{3}x^3\right]_0^{1.2} = \frac{1}{8} \times (1.2^3 - 0^3) = 0.216$$

$$P(X < 1.2) = F(1.2) = \frac{1.2^3}{8} = 0.216$$

Example

$$f(x) = \begin{cases} \frac{3}{8}x^2 \ for\ 0 \le x \le 2 \\ 0 \qquad\qquad elsewhere \end{cases}$$

$$F(x) = \begin{cases} 0 & for \quad x < 0 \\ \dfrac{x^3}{8} & for\, 0 \le x \le 2 \\ 1 & for \quad x > 2 \end{cases}$$

$$P(1 < X < 1.5) = \int_1^{1.5} \frac{3}{8}x^2 dx = \frac{3}{8} \times \left[\frac{1}{3}x^3\right]_1^{1.5} = \frac{1}{8} \times (1.5^3 - 1^3) = 0.297$$

$$P(1 < X < 1.5) = F(1.5) - F(1) = \frac{1.5^3}{8} - \frac{1^3}{8} = 0.297$$

Using cumulative distribution function

$$P(X \leq x) = F(x)$$

$$P(X > x) = 1 - F(x)$$

$$P(a \leq X \leq b) = F(b) - F(a)$$

Expected value

$$E(X) = \int_{-\infty}^{\infty} x \times f(x)dx$$

$$E(aX + b) = aE(X) + b$$

$$E(X + Y) = E(X) + E(Y)$$

Expected value

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$f(x) = \begin{cases} \frac{3}{8} x^2 \; for \; 0 \leq x \leq 2 \\ 0 \qquad \qquad elsewhere \end{cases}$$

$$E(X) = \int_0^2 x \;\; \frac{3}{8} x^2 dx = \frac{3}{8} \int_0^2 x^3 dx = \frac{3}{8} \times \left[\frac{1}{4} x^4 \right]_0^2 =$$

$$= \frac{3}{32} (2^4 - 0) = \frac{3}{32} \times 16 = \frac{3}{2}$$

Expected value of a function of random variable

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) \times f(x)dx$$

In particular

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \times f(x)dx$$

$$Var(X) = E(X^2) - (E(X))^2$$

Example

$$f(x) = \begin{cases} \frac{3}{8}x^2 \; for \; 0 \le x \le 2 \\ 0 \qquad\qquad elsewhere \end{cases}$$

$$E(X^2) = \int_0^2 x^2 \; \frac{3}{8}x^2 dx = \frac{3}{8}\int_0^2 x^4 dx = \frac{3}{8} \times \left[\frac{1}{5}x^5\right]_0^2 =$$

$$= \frac{3}{40}(2^5 - 0) = \frac{3}{40} \times 32 = \frac{12}{5}$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{12}{5} - \left(\frac{3}{2}\right)^2 = \frac{3}{20}$$

Example

$$f(x) = \begin{cases} \frac{3}{8}x^2 \; for \; 0 \le x \le 2 \\ 0 \qquad\qquad elsewhere \end{cases}$$

$$E(X^3) = \int_0^2 x^3 \; \frac{3}{8}x^2 dx = \frac{3}{8}\int_0^2 x^5 dx = \frac{3}{8} \times \left[\frac{1}{6}x^6\right]_0^2 =$$

$$= \frac{1}{16}(2^6 - 0) = \frac{1}{16} \times 64 = 4$$

$$E(X^4) = \int_0^2 x^4 \; \frac{3}{8}x^2 dx = \frac{3}{8}\int_0^2 x^6 dx = \frac{3}{8} \times \left[\frac{1}{7}x^7\right]_0^2 =$$

$$= \frac{3}{56}(2^7 - 0) = \frac{3}{56} \times 128 = 6.86$$

$$f(x) = \begin{cases} \frac{2}{x^2} & for\ x \geq 2 \\ 0 & elsewhere \end{cases}$$

$$P(X > 6) = \int_6^\infty \frac{2}{x^2} dx = 2\int_6^\infty x^{-2} dx = 2 \times \left[\frac{1}{-1} x^{-1}\right]_6^\infty =$$

$$= -2 \times \left[\frac{1}{x}\right]_6^\infty = -2\left(0 - \frac{1}{6}\right) = \frac{1}{3}$$

$$f(x) = \begin{cases} \dfrac{2}{x^2} & for\ x \geq 2 \\ 0 & elsewhere \end{cases}$$

$$P(X \leq 6) = \int_2^6 \frac{2}{x^2} dx = 2\int_2^6 x^{-2} dx = 2 \times \left[\frac{1}{-1} x^{-1}\right]_2^6 =$$

$$= -2 \times \left[\frac{1}{x}\right]_2^6 = -2\left(\frac{1}{6} - \frac{1}{2}\right) = -2\left(\frac{2}{12} - \frac{6}{12}\right) = \frac{2}{3}$$

$$P(X > 6) = 1 - \frac{2}{3} = \frac{1}{3}$$

Common types of continuous probability distributions

Continuous uniform distribution

Exponential distribution

Normal distribution

Continuous uniform distribution

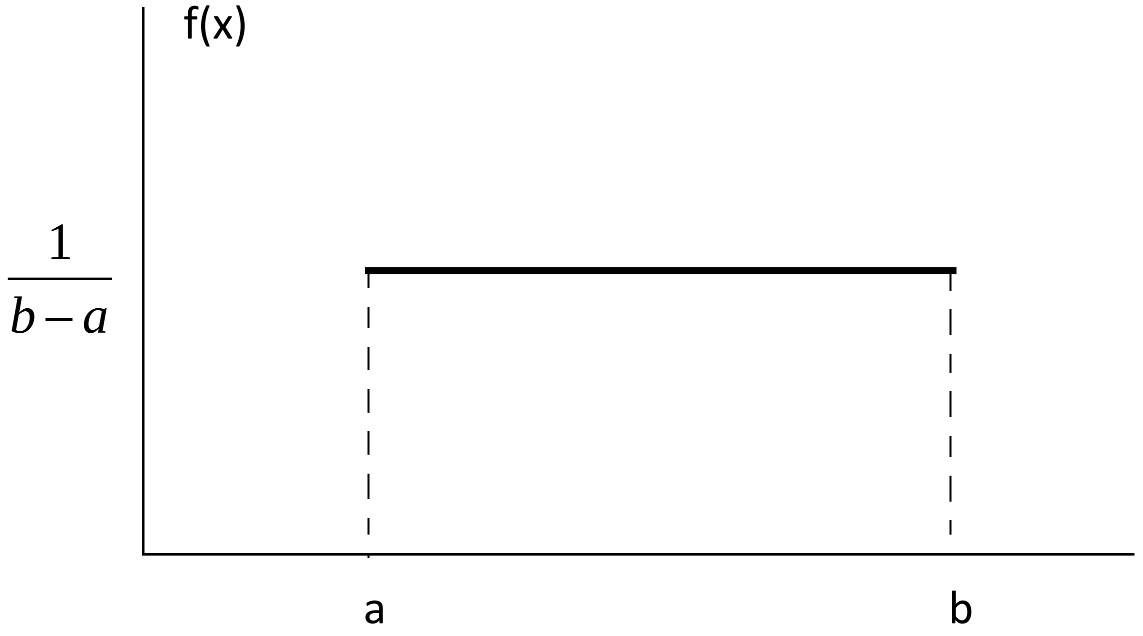
If X has continuous uniform distribution in an interval (a,b) then its probability density function is

$$f(x) = \frac{1}{b-a}$$

for a < x < b and f(x) = 0 otherwise.

Continuous uniform distribution

Probability density function is constant:



Area under the curve = area of the rectangle = $(b-a)\times\frac{1}{b-a}=1$

Continuous uniform distribution

Cumulative distribution function can be expressed with a formula:

$$F(x) = \begin{cases} 0 & for \quad x < a \\ \dfrac{x-a}{b-a} & for\, a \le x \le b \\ 1 & for \quad x > b \end{cases}$$

Continuous uniform distribution

- Expected value

$$E(X) = \frac{a+b}{2}$$

- Variance

$$Var(X) = \frac{(b-a)^2}{12}$$

Example

X has a uniform distribution in an interval (0,10). Calculate $P(X<2)$, $P(X>5)$ and $P(3<X<7)$.

Calculate $E(X)$ and $Var(X)$.

Solution

X has uniform distribution with a = 0 and b = 10. Then

$$f(x) = \frac{1}{10}\ for\ 0 < x < 10$$

$$P(X < 2) = \int_0^2 \frac{1}{10} dx = \left[\frac{1}{10}x\right]_0^2 = \frac{1}{10} \times (2 - 0) = \frac{2}{10} = 0.2$$

$$P(X > 5) = \int_5^{10} \frac{1}{10} dx = \left[\frac{1}{10}x\right]_5^{10} = \frac{1}{10} \times (10 - 5) = \frac{5}{10} = 0.5$$

Solution

X has uniform distribution with a = 0 and b = 10. Then

$$f(x) = \frac{1}{10} \, for \; 0 < x < 10$$

$$P(3 < X < 7) = \int_3^7 \frac{1}{10} dx = \left[\frac{1}{10} x\right]_3^7 = \frac{1}{10} \times (7 - 3) = \frac{4}{10} = 0.4$$

Solution

X has uniform distribution with a = 0 and b = 10. Then

$$F(x) = \begin{cases} 0 & for \quad x < 0 \\ \dfrac{x}{10} & for\, 0 \le x \le 10 \\ 1 & for \quad x > 10 \end{cases}$$

$$P(X < 2) = F(2) = \frac{2}{10} = 0.2$$

Solution

$$P(X > 5) = 1 - F(5) = 1 - \frac{5}{10} = \frac{1}{2} = 0.5$$

$$P(3 < X < 7) = F(7) - F(3) = \frac{7}{10} - \frac{3}{10} = \frac{4}{10} = 0.4$$

$$E(X) = \frac{0+10}{2} = 5$$

$$Var(X) = \frac{(10-0)^2}{12} = \frac{100}{12} = \frac{25}{3} = 8.33$$